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## LETTER TO THE EDITOR

**Commutation relations for linear fields: a coordinate-free approach**

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**Abstract.** The commutation relations for a linear system are given directly in terms of the classical energy and equations of motion, without the need for a normal mode expansion. The formula is illustrated in the case of the free electromagnetic field.

The commonest methods of deriving the commutation relations for fields are probably to decompose into normal modes or to find canonical coordinates and momenta, and both methods depend in some measure on a choice of coordinates. For linear systems the symplectic form  $s$  provides a coordinate-free alternative [3, 4]. (This is just the Poisson bracket applied to linear functions on phase space  $V$ , and can be characterized as an antisymmetric bilinear form on  $V$  having maximal rank.) Each  $a \in V$  defines a linear function (the moment map) on phase space  $\phi_a(v) = s(a, v)$ , and every linear function is of this form. The Poisson bracket of two such functions is given by  $\{\phi_a, \phi_b\} = s(a, b)$ , giving us the Dirac quantization rule

$$[\mathcal{Q}(\phi_a), \mathcal{Q}(\phi_b)] = i\hbar s(a, b) \quad (1)$$

for the quantized observables  $\mathcal{Q}(\phi_a)$  and  $\mathcal{Q}(\phi_b)$ . Unfortunately, although geometrically natural, the symplectic form is often not immediately known, and must first be calculated from the energy and equations of motion. The purpose of this letter is to point out that a slight modification leads to a more efficient procedure for finding the symplectic structure and commutation relations at the same time. The difference will be illustrated in the case of the free electromagnetic field.

In classical Hamiltonian mechanics the trio of Poisson bracket, Hamiltonian  $H_{cl}$  and time evolution are related by the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H_{cl}, f\} \quad (2)$$

and each member of the trio can be recovered from the other two. In practice we most readily know the time evolution equations and the Hamiltonian, which for these linear systems is just the energy,  $E$ . The energy of a linear system is a quadratic function on phase space, from which we may easily derive the symmetric bilinear form

$$\mathcal{E}(u, v) = E(u + v) - E(u) - E(v). \quad (3)$$

(We have  $\mathcal{E}(u, u) = E(2u) - 2E(u) = 4E(u) - 2E(u) = 2E(u)$ , and this determines  $\mathcal{E}$  uniquely.) The positivity of  $E$  means that  $\mathcal{E}$  actually defines a real inner product on phase space. The linear functions  $\mathcal{E}_u(v) = \mathcal{E}(u, v)$  define classical phase space observables in

terms of which we can give the commutation relations. Before doing so, however, we need to introduce the equations of motion. For a linear system these can be given directly as a differential equation for the phase space trajectory of the form

$$\frac{dv}{dt} = \Omega v \quad (4)$$

where  $v \in V$  and  $\Omega$  is a linear operator on  $V$ .

Hamilton's equations of motion for linear systems may be expressed as

$$s(\Omega u, v) = \mathcal{E}(u, v) \quad (5)$$

(see [2] equation (4.1) or, for a direct proof, the appendix.) This means that  $\mathcal{E}_u = \phi_{\Omega u}$ , so that

$$\{\mathcal{E}_u, \mathcal{E}_v\} = \{\phi_{\Omega u}, \phi_{\Omega v}\} = s(\Omega u, \Omega v) = \mathcal{E}(u, \Omega v) \quad (6)$$

and Dirac's quantization condition now gives us the commutation relation

$$[\mathcal{Q}(\mathcal{E}_u), \mathcal{Q}(\mathcal{E}_v)] = i\hbar \mathcal{E}(u, \Omega v). \quad (7)$$

We shall now give some examples to show how easily this leads to the commutation relations.

### Example 1

As a first illustration we first apply this result to a one-dimensional oscillator, with energy

$$E(x, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \quad (8)$$

We immediately calculate that

$$\mathcal{E}((x_1, p_1), (x_2, p_2)) = m^{-1}p_1 p_2 + m\omega^2 x_1 x_2. \quad (9)$$

For  $a = (a_p, a_x)$ , we have the linear function

$$\mathcal{E}_a(x, p) = m^{-1}a_p p + m\omega^2 a_x x. \quad (10)$$

The equations of motion,  $\dot{x} = m^{-1}p$ ,  $\dot{p} = -m\omega^2 x$ , can be expressed in the matrix form

$$\frac{d}{dt} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} m^{-1}p \\ -m\omega^2 x \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -m\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad (11)$$

so that

$$\Omega = \begin{pmatrix} 0 & m^{-1} \\ -m\omega^2 & 0 \end{pmatrix}. \quad (12)$$

Combining these, and writing  $P = \mathcal{Q}(p)$  and  $X = \mathcal{Q}(x)$  for the quantized momentum and position, the commutation relation (7) becomes

$$[m^{-1}a_p P + m\omega^2 a_x X, m^{-1}b_p P + m\omega^2 b_x X] = i\hbar(a_p(-\omega^2 b_x) + \omega^2 a_x b_p) \quad (13)$$

or

$$\omega^2(a_x b_p - a_p b_x)[X, P] = i\hbar\omega^2(a_x b_p - a_p b_x). \quad (14)$$

In other words we recover the normal commutation relations between position and momentum. One can easily extend this example to cover arbitrary finite-dimensional oscillators.

## Example 2

This method is particularly useful for systems such as the free electromagnetic field, for which there are no obvious coordinates and momenta. The energy of the electromagnetic field is given by

$$E(\mathbf{E}, \mathbf{B}) = \frac{1}{2} \int (\epsilon_0 |\mathbf{E}|^2 + \mu_0^{-1} |\mathbf{B}|^2) d^3x \quad (15)$$

from which we obtain the bilinear form

$$\mathcal{E}((e_1, \mathbf{b}_1), (e_2, \mathbf{b}_2)) = \int (\epsilon_0 e_1 \cdot e_2 + \mu_0^{-1} \mathbf{b}_1 \cdot \mathbf{b}_2) d^3x \quad (16)$$

on vector-valued test functions  $e_k$  and  $\mathbf{b}_k$ . (Phase space consists of pairs of such functions  $(e, \mathbf{b})$  having finite energy.)

Maxwell's equations for the electromagnetic field give

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl } \mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{B}}{\partial t} = -\text{curl } \mathbf{E} \quad (17)$$

so that

$$\frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & c^2 \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \quad (18)$$

from which we obtain

$$\Omega = \begin{pmatrix} 0 & c^2 \text{curl} \\ -\text{curl} & 0 \end{pmatrix}. \quad (19)$$

Then the commutation relations take the form

$$[\mathcal{Q}(\mathcal{E}_{(e_1, \mathbf{b}_1)}), \mathcal{Q}(\mathcal{E}_{(e_2, \mathbf{b}_2)})] = \mathcal{E}((e_1, \mathbf{b}_1), (c^2 \text{curl } \mathbf{b}_2, -\text{curl } e_2)) \quad (20)$$

$$= i\hbar \int \mu_0^{-1} (e_1 \cdot \text{curl } \mathbf{b}_2 - \mathbf{b}_1 \cdot \text{curl } e_2) d^3x. \quad (21)$$

Writing

$$\mathcal{Q}(\mathcal{E}_{(e,0)}) = \int \epsilon_0 \hat{\mathbf{E}}(\mathbf{x}) \cdot e(\mathbf{x}) d^3x \quad \text{and} \quad \mathcal{Q}(\mathcal{E}_{(0,\mathbf{b})}) = \int \mu_0^{-1} \hat{\mathbf{B}}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}) d^3x \quad (22)$$

we deduce that the components of the quantized electric field  $\hat{\mathbf{E}}$  commute amongst themselves as do those of the quantized magnetic field  $\hat{\mathbf{B}}$ , but that the electric and magnetic fields do not commute with each other. Indeed, letting  $\mathbf{b}_1$  and  $e_2$  vanish and dropping the suffices on the other two test fields we have

$$\left[ \int \epsilon_0 \hat{\mathbf{E}}(\mathbf{x}) \cdot e(\mathbf{x}) d^3x, \int \mu_0^{-1} \hat{\mathbf{B}}(\mathbf{y}) \cdot \mathbf{b}(\mathbf{y}) d^3y \right] = i\hbar \int \mu_0^{-1} e \cdot \text{curl } \mathbf{b} d^3x \quad (23)$$

from which the usual fixed time commutation relations,

$$[\hat{E}_j(\mathbf{x}), \hat{B}_k(\mathbf{y})] = -i\hbar \mu_0 c^2 \epsilon_{jkl} \delta_{,l}(\mathbf{x} - \mathbf{y}) \quad (24)$$

follow immediately.

We can also see that, for any  $f$ ,  $\hat{\mathbf{B}}(\text{grad } f)$  commutes with all the fields. Moreover, a formal use of the divergence theorem gives

$$\hat{\mathbf{B}}(\text{grad } f) = \int \hat{\mathbf{B}}(\mathbf{x}) \cdot \text{grad } f(\mathbf{x}) d^3x = - \int \text{div } \hat{\mathbf{B}}(\mathbf{x}) f(\mathbf{x}) d^3x \quad (25)$$

so that  $\text{div } \hat{\mathbf{B}}$  commutes with all fields. This permits us to impose the additional constraint that  $\text{div } \hat{\mathbf{B}} = 0$ . Similarly we can impose the Gauss constraint  $\text{div } \hat{\mathbf{E}} = 0$

We conclude by noting that, in this case, the identity  $\phi_{\Omega u} = \mathcal{E}_u$  becomes

$$\phi_{(c^2 \text{curl } \mathbf{b}, -\text{curl } \mathbf{e})} = \mathcal{E}_{(e, \mathbf{b})}. \quad (26)$$

Not only is an integration required to obtain an explicit formula for  $\phi_{(e, \mathbf{b})}$ , but the resulting expression in terms of Green's functions is non-local, which somewhat obscures the meaning of the resulting commutation relation for the  $\mathcal{Q}(\phi_{(e, \mathbf{b})})$ . By contrast, the commutation relations for  $\mathcal{Q}(\mathcal{E}_{(e, \mathbf{b})})$  are manifestly local and transparent.

### Example 3

Consider electromagnetic radiation interacting with a continuum of dipoles as in the model of a light polarization measurement presented in [1]. Maxwell's equations relating the classical electric and magnetic fields,  $\mathbf{e}$  and  $\mathbf{b}$ , with the dielectric polarization vector  $\mathbf{p}$  and its derivative  $\mathbf{j} = \partial \mathbf{p} / \partial t$ , give

$$\frac{\partial \mathbf{e}}{\partial t} = c^2 \text{curl } \mathbf{b} - \epsilon_0^{-1} \mathbf{j} \quad \frac{\partial \mathbf{b}}{\partial t} = -\text{curl } \mathbf{e}. \quad (27)$$

The polarization vector is thought of as the sum of the molecular dipole moments, and restoring and damping forces give an equation of the form

$$\frac{\partial \mathbf{j}}{\partial t} + \eta \mathbf{j} + \omega_0^2 \mathbf{p} = v^{-1} \mathbf{e}. \quad (28)$$

For the present purposes we shall ignore the damping term and  $\eta = 0$ . The Hamiltonian for the system is given by

$$E = \frac{1}{2} \int (\epsilon_0 (|\mathbf{e}|^2 + c^2 |\mathbf{b}|^2) + v (|\mathbf{j}|^2 + \omega_0^2 |\mathbf{p}|^2)) dV. \quad (29)$$

In equilibrium when  $\mathbf{j} = 0$  and  $\mathbf{p} = \mathbf{e} / v \omega_0^2$ , with  $\mathbf{e}$  and  $\mathbf{b}$  constant, this reduces to the usual formula for the dielectric energy.

We obtain the following commutation relations from the equations of motion and formula for the energy,

$$\begin{aligned} \frac{i}{\hbar} [\mathcal{Q}(e_1, \mathbf{b}_1, \mathbf{p}_1, \mathbf{j}_1), \mathcal{Q}(e_2, \mathbf{b}_2, \mathbf{p}_2, \mathbf{j}_2)] &= \int \mu_0^{-1} (\text{curl } \mathbf{b}_1 \cdot \mathbf{e}_2 - \text{curl } \mathbf{e}_1 \cdot \mathbf{b}_2) dV \\ &+ \int (\mathbf{e}_1 \cdot \mathbf{j}_2 - \mathbf{e}_2 \cdot \mathbf{j}_1) dV + \int v \omega_0^2 (\mathbf{j}_1 \cdot \mathbf{p}_2 - \mathbf{j}_2 \cdot \mathbf{p}_1) dV. \end{aligned}$$

### Appendix. The relation between the energy and the symplectic form

For completeness we sketch a proof of the crucial relationship  $s(\dot{u}, v) = s(\Omega u, v) = \mathcal{E}(u, v)$ , used earlier. Since  $u$  and  $v$  are made up out of linear coordinate functions, the definition of  $s$  together with Hamilton's equations give us

$$s(\dot{u}, v) = \{\dot{u}, v\} = \{\{H_{cl}, u\}, v\}. \quad (30)$$

As  $H_{cl} = E$ , the right-hand side, which can be rearranged as  $\{v, \{u, H_{cl}\}\}$ , is just the second derivative  $D_v D_u E$  of the energy. Now the directional (Fréchet) derivative of the energy is

$$(D_u E)(w) = \frac{d}{ds} E(w + su)|_{s=0} = \frac{d}{ds} (E(w) + s\mathcal{E}(w, u) + s^2 E(u))|_{s=0} = \mathcal{E}(u, w) \quad (31)$$

and a second derivative is

$$(D_v D_u E)(w) = \frac{d}{ds} \mathcal{E}(u, w + sv)|_{s=0} = \mathcal{E}(u, v) \quad (32)$$

so that  $s(\Omega u, v) = \{v, \{u, H_{cl}\}\} = \mathcal{E}(u, v)$ , as we claimed.

One can follow a similar route with nonlinear systems and quantize classical functions  $\phi_{\{H,u\}}$ , but, as there is no longer a distinguished set of linear variables  $u$ , the result is somewhat arbitrary.

## References

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